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The Stable Manifold of a Point of a Hyperbolic Map of a Banach Space*

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In [1], Hirsch and Pugh announce a theorem on the stable manifold of a "nearly fixed" point of a hyperbolic map of a Banach space into itself. (Details will be given in [2].) The object of this note is to give a simple proof of their theorem, following the proof in [3, Exercise 5.4, pp. 241, 566–567], for the case of a fixed point in Euclidean space. By weakening "hyperbolicity" slightly, it will be seen that there is no loss of generality in supposing that the "nearly fixed" point is actually a "fixed" point. This weakening of hyperbolicity also facilitates the handling of parameters and proofs of differentiability. Our results concern, not the iterates $T \circ \cdots \circ T$ of a fixed map T but, more generally, the products $T_n \circ T_{n-1} \circ \cdots \circ T_1$ of a given sequence of maps $T_1, T_2, \ldots$. This generalization also has methodical advantages.

The "existence and continuity" proof is suggested by a procedure of Hartman and Wintner [4, 5] in the asymptotic theory of ordinary differential equations, and employed by Coffman [6] in dealing with difference equations in a context similar to the situation here. "Differentiability" will be deduced from "continuity and continuous dependence on parameters," as in Hadamard's proof for the smooth dependence of solutions of ordinary differential equations on initial conditions and parameters; cf. [3, pp. 95–100]. The main results are stated in Section 3; see also Section 7.

An appendix deals with stable manifolds of Anosov diffeomorphisms.

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1. Notation and Hypotheses

Let $W, X, Y, Z = X \times Y = XY$, $E = W \times X \times Y = WXY$ be Banach spaces. Norms for $w \in W, x \in X, y \in Y, z = (x, y) \in Z, e = (w, x, y) \in E$ will be denoted by

$$\| x \|, \| y \|, \| z \| = \max(\| x \|, \| y \|),$$
$$\| e \| = \max(\| w \|, \| x \|, \| y \|).$$

Finally, let $X_r(x_0) = \{x \in X : \| x - x_0 \| < r\}$.

Definition Stability Set. Let $X^0 \subset X$ be open; $T_n' : X^0 \to X$ a map for $n = 1, 2, ...$; $\mathscr{D}(T_n' \circ \cdots \circ T_1')$ the (possibly empty) open set where $T_n' \circ \cdots \circ T_1'$ is defined. The (possibly empty) subset $\mathscr{D} = \mathscr{D}(\{T_n'\}, X^0) = \bigcap \mathscr{D}(T_n' \circ \cdots \circ T_1')$ of X^0 will be called the *stability set* of the sequence $\{T_n'\}$, with respect to X^0.

We shall be interested in stability sets of sequences of maps $T_n' : Z_1(0) \to Z$ of the form

$$T_n'(z) = (A_n x, B_n y) + (F_n'(z) + x^n, G_n'(z) + y^n), \tag{1.0}$$

where $A_n \in L(X, X)$, $B_n \in L(Y, Y)$ are bounded linear operators, B_n is invertible, $\| A_n \| < 1$, $\| B^{-1} \| < 1$; F_n' and G_n' are maps from $Z_1(0)$ to X and Y, respectively, such that $F_n'(0) = 0$, $G_n'(0) = 0$; $\| F_n'(z_1) - F_n'(z_2) \|$, $\| G_n'(z_1) - G_n'(z_2) \| \leqslant \epsilon \| z_1 - z_2 \|$; and $\| x^n \|, \| y^n \| \leqslant \epsilon$; where $\epsilon > 0$ is a fixed small number. In order to reduce (1.0) to the case $x^n = 0$, $y^n = 0$, we can replace Z, T_n' by $R^1 Z, T_n''$, where

$$T_n''(t, z) = (t, A_n x, B_n y) + (0, F_n'(z) + 2tx^n, G_n'(z) + 2ty^n). \tag{1.1}$$

It is clear that the stable set $\mathscr{D}(\{T_n'\}, Z_1(0))$ will be determined if $\mathscr{D}(\{T_n''\}, (R^1 Z)_1(0))$ is known; in fact,

$$(x, y) \in \mathscr{D}(\{T_n'\}, Z_1(0)) \Leftrightarrow (1/2, x, y) \in \mathscr{D}(\{T_n''\}, (R^1 Z)_1(0)).$$

This reduction to the case $t^n = 0$, $x^n = 0$, $y^n = 0$ has been accomplished at the cost of replacing the linear map $x \mapsto A_n x$, having a norm $a < 1$, by a map $(t, x) \mapsto (t, A_n x)$, having norm 1. It will turn out that, in view of the form of the latter, nothing is lost and, indeed, there are other advantages.

For these "methodical" reasons, we shall not consider maps (1.0) below, but maps $T_n : E_r(0) \to E = W \times X \times Y = WXY$ of the form

$$T_n(e) = (U_n(w), A_n x, B_n y) + (0, F_n(e), G_n(e)), \tag{1.2}$$

where r is fixed $(0 < r < \infty)$; $A_n \in L(X, X)$ and $B_n \in L(Y, Y)$ are linear bounded operators, B_n is invertible,

$$\| A_n \| \leqslant a < 1, \qquad \| B_n^{-1} \| \leqslant 1/b < 1; \tag{1.3}$$

$$0 < \delta < 1/10, \qquad 0 < a < 1 - 2\delta < 1 + 5\delta < b; \tag{1.4}$$

$$U_n(0) = 0, \qquad F_n(0) = 0, \qquad G_n(0) = 0; \tag{1.5}$$

$U_n : W_r(0) \to W$; F_n, $G_n : E_r(0) \to X, Y$; and

$$\| U_n(w_1) - U_n(w_2) \| \leqslant \| w_1 - w_2 \|, \tag{1.6}$$

$$\| F_n(e_1) - F_n(e_2) \|, \| G_n(e_1) - G_n(e_2) \| \leqslant \delta^2 \| e_1 - e_2 \|. \tag{1.7}$$

The stability set of $\{T_n\}$ with respect to $\{E_r(0)\}$ is

$$\mathscr{D} = \bigcap_{n=1}^{\infty} \mathscr{D}_n, \qquad \mathscr{D}_n = \mathscr{D}(S_n), \qquad S_n = T_n \circ \cdots \circ T_1 . \tag{1.8}$$

The first result (Section 2) will give, under suitable assumptions on U_n, F_n and G_n, the existence of a continuous function $y_0 : (WX)_r(0) \to Y_r(0)$ such that

$$\mathscr{D} = \{e = (w, x, y) : y = y_0(w, x) \text{ on } (WX)_r(0)\}. \tag{1.9}$$

In such a case, we shall speak of the *stable manifold* $\mathscr{D} : y = y_0(w, x)$ of the sequence $\{T_n\}$, with respect to $E_r(0)$. Actually, the assumptions on U_n, F_n, and G_n will imply that $y_0 \in C^1$, but this and other smoothness properties will be deduced from the existence proof for a continuous $y_0(w, x) = y_0(w, x, \sigma)$ carried out for the case that T_n depends on parameters σ.

HYPOTHESIS (H). Let Σ be a metric space and $\sigma \in \Sigma$. Let $T_n \in C^0(E_r(0) \times \Sigma, E)$. For each $\sigma \in \Sigma$, assume that T_n satisfies (1.2)–(1.7), where a, b, δ are independent of σ. Let T_n have a Fréchet derivative $D_e T_n \in C^0(E_r(0) \times \Sigma)$.

HYPOTHESIS (H_0). Let $\Sigma = \Lambda \times U \times T$, where Λ, U are metric spaces and $T = [0, \epsilon]$ is a real t-interval. We say that T_1, T_2,... satisfies (H_0) if (H) holds, $T_{n0} = (T_n)_{t=0}$ is independent of $u \in U$, and T_n, $D_e T_n \to T_{n0}$, $D_e T_{n0}$ uniformly on U, for fixed (e, λ), as $t \to +0$.

By (1.4), we have

$$\alpha \equiv b - \delta/(1 - 2\delta) > (1 + \delta)/(1 - 2\delta) > 1; \tag{1.10}$$

also, if

$$c_n = \sum_{k=0}^{n} (a + \delta^2)^k \qquad \text{and} \qquad c = \lim_{h\to\infty} c_n = 1/(1 - a - \delta^2), \quad (1.11)$$

then $a < 1 - 2\delta < 1 - \delta - \delta^2$ implies that

$$\delta^2 c_n < \delta^2 c < \delta. \tag{1.12}$$

Generally, we shall not exhibit the dependence of functions on parameters, unless convenient to do so.

2. Main Lemma

In this section, we prove

Lemma 2.1(a). *Assume hypothesis* (H). *Then there exists* $y_0(w, x) = y_0(w, x, \sigma) \in C^0((WX)_r(0) \times \Sigma, Y_r(0))$ *such that, for a fixed* σ, *the stability set* $\mathscr{D} = \mathscr{D}(\{T_n\}, E_r(0))$ *is given by* (1.9); *also if* $e_0 = (w_0, x_0, y_0(w_0, x_0))$ *and* $e_k = S_k(e_0) = (w_k, x_k, y_k)$, *then, for* $k \geqslant 0$, $e_k = 0$ *if* $(w_0, x_0) = 0$,

$$\begin{aligned}\| y_k(w^0, x^0) - y_k(w_0, x_0)\| \\ \leqslant (1 - 2\delta)(\delta \| w^0 - w_0 \| + \| x_k(w^0, x^0) - x_k(w_0, x_0)\|),\end{aligned} \tag{2.1}$$

$$\| x_k(w^0, x^0) - x_k(w_0, x_0)\| \leqslant \delta \| w^0 - w_0 \| + (a + 2\delta^2)^k \| x^0 - x_0 \|; \tag{2.2}$$

and the manifold $y = y_0(w, x)$ *is "invariant," that is,*

$$y_k(w_0, x_0) \equiv y_0(w_k(w_0, x_0), x_k(w_0, x_0)) \quad \text{on} \quad (WX)_r(0). \tag{2.3}$$

(b) *If, in addition,* (H_0) *holds and* $\sigma = (\lambda, u, t)$, *then* $y_0(w, x, \lambda, u, 0)$ *is independent of* u *and* $y_0(w, x, \lambda, u, t) \to y_0(w, x, \lambda, u, 0)$ *uniformly on* U, *for fixed* (w, x, λ), *as* $t \to +0$.

Note that (2.1), (2.2) imply

$$\| y_k(w, x)\| \leqslant (1 - 2\delta)(\delta \| w \| + \| x_k(w, x)\|) < (1 - 4\delta^2)r \text{ for } k \geqslant 0, \tag{2.1$'$}$$

$$\| x_k(w, x)\| \leqslant \delta \| w \| + (a + 2\delta^2)^k \| x \| < (a + 2\delta)r \qquad \text{for} \quad k \geqslant 1, \tag{2.2$'$}$$

where $w = w_0$, $x = x_0$. But if $y \neq y_0(w, x)$ and $e_0 = (w, x, y) \in \mathscr{D}_n$ for some $n > 0$, then $e_k = S_k(e_0) = (w_k, x_k, y_k)$ satisfies

$$\| y_k \| \geqslant \alpha^k \| y_{0k}(w, x) - y \| \qquad \text{for} \quad k = 0,\dots, n, \tag{2.2$''$}$$

where (2.18) holds; cf. the proof of Proposition 2.5. In particular, if $\alpha^N \| y_{0N}(w, x) - y \| \geqslant r$, then $(w, x, y) \notin \mathscr{D}_N$.

PROPOSITION 2.1. *Let Γ be a complete metric space, and Σ as in* (H); *$C = C(\gamma, \sigma) : \Gamma \times \Sigma \to \Gamma$ continuous and, for fixed σ, $C_\sigma = C(\cdot, \sigma) : \Gamma \to \Gamma$ a contraction*

$$\operatorname{dist}(C_\sigma(\gamma^0), C_\sigma(\gamma_0)) \leqslant \theta \operatorname{dist}(\gamma^0, \gamma_0), \qquad 0 \leqslant \theta < 1,$$

with θ independent of σ. Then, for each $\sigma \in \Sigma$, the map $C_\sigma : \Gamma \to \Gamma$ has a unique fixed point $\gamma = \gamma(\sigma)$ and $\gamma(\sigma) \in C^0(\Sigma, \Gamma)$. If $\Sigma = \Lambda \times U \times T$ as in (H_0), *$C(\gamma, \lambda, u, 0)$ is independent of $u \in U$, and $C(\gamma, \lambda, u, t) \to C(\gamma, \lambda, u, 0)$, as $t \to +0$, uniformly on U for fixed (γ, λ), then $\gamma(\lambda, u, 0)$ is independent of u and $\gamma(\lambda, u, t) \to \gamma(\lambda, u, 0)$ uniformly on U, for fixed λ, as $t \to +0$.*

For, by the proof of the contraction principle, if $\gamma_0 \in \Gamma$, then $\gamma(\sigma) = \lim C_\sigma^{\,n}(\gamma_0)$, as $n \to \infty$, and $\operatorname{dist}(\gamma(\sigma), C_\sigma^{\,n}(\gamma_0)) \leqslant \theta^n \operatorname{dist}(C_\sigma(\gamma_0), \gamma_0)/(1 - \theta)$.

PROPOSITION 2.2. *Let $n > 0$; $e_0, e^0 \in \mathscr{D}_n = \mathscr{D}(S_n)$; and $e_k = S_k(e_0)$, $e^k = S_k(e^0)$ for $k = 0, \ldots, n$.*

(a) *The inequality*

$$\| y^m - y_m \| \geqslant (1 - 2\delta)(\delta \| w^m - w_m \| + \| x^m - x_m \|) \quad \text{for some } m, \quad 0 \leqslant m < n \tag{2.4}$$

(*for example, $w^m = w_m$, $x^m = x_m$), implies that, for $k = m + 1, \ldots, n$,*

$$\begin{aligned} \| y^k - y_k \| &\geqslant \delta \| w^k - w_k \| + \| x^j - x_k \|, \\ \| y^k - y_k \| &\geqslant \alpha^{k-m} \| y^m - y_m \|, \end{aligned} \tag{2.5}$$

where $\alpha > 1$; cf. (1.10).

(b) *The inequality*

$$\| y^n - y_n \| < (1 - 2\delta)(\delta \| w^n - w_n \| + \| x^n - x_n \|) \tag{2.6}$$

(*for example, $y^n = y_n$) and* (1.11), (1.12) *imply that, for $k = 0, \ldots, n$,*

$$\| y^k - y_k \| \leqslant (1 - 2\delta)(\delta \| w^k - w_k \| + \| x^k - x_k \|), \tag{2.7}$$

$$\| x^k - x_k \| \leqslant \delta^2 c_k \| w^0 - w_0 \| + (a + \delta^2)^k \| x^0 - x_0 \|, \ \delta^2 c_k < \delta. \tag{2.8}$$

Proof (a). Since $\| e \| = \max(\| w \|, \| x \|, \| y \|)$, (2.4) implies that

$$\delta \| e^m - e_m \| \leqslant \| y^m - y_m \| / (1 - 2\delta). \tag{2.9}$$

Hence, by (1.6) and (1.7), $\| w^{m+1} - w_{m+1} \| \leqslant \| w^m - w_m \|$ and

$$\| x^{m+1} - x_{m+1} \| \leqslant a \| x^m - x_m \| + \delta \| y^m - y_m \|/(1 - 2\delta);$$

so that, by (2.4),

$$\delta \| w^{m+1} - w_{m+1} \| + \| x^{m+1} - x_{m+1} \| \leqslant (1 + \delta) \| y^m - y_m \|/(1 - 2\delta).$$

Also, (2.9) and (1.6), (1.7) give

$$\| y^{m+1} - y_{m+1} \| \geqslant [b - \delta/(1 - 2\delta)] \| y^m - y_m \|. \qquad (2.10)$$

The last two inequalities and (1.10) give (2.5) for $k = m + 1$, and an induction gives it for $k = m + 1, \ldots, n$.

Proof (b). The inequalities (2.7) follow from part (a). In order to obtain (2.8), note that (2.7) implies that

$$\| e^k - e_k \| \leqslant \| w^k - w_k \| + \| x^k - x_k \| \leqslant \| w^0 - w_0 \| + \| x^k - x_k \|.$$

Hence, by (1.7),

$$\| x^{k+1} - x_{k+1} \| \leqslant \delta^2 \| w^0 - w_0 \| + (a + \delta^2) \| x^k - x_k \|.$$

An induction gives (2.8).

PROPOSITION 2.3. *Write $S_n(e) = (P_n(e), Q_n(e), R_n(e))$, where $e \in \mathscr{D}_n$ and $P_n \in W$, $Q_n \in X$, $R_n \in Y$. For a fixed σ and $n \geqslant 1$, there exists a function*

$$y_{0n}(w, x) = y_{0n}(w, x, \sigma) \in C^0((WX)_r(0) \times \Sigma, Y_r(0))$$

such that $(w, x, y_{0n}(w, x)) \in \mathscr{D}_n$; $y_{0n} = 0$ if $(w, x) = 0$; and

$$R_n(w, x, y) = 0 \quad \textit{and only if} \quad y = y_{0n}(w, x). \qquad (2.11)$$

If, in addition (H_0) *holds, then y_{0n} is independent of u when $t = 0$ and $y_{0n}(w, x, \lambda, u, t) \to y_{0n}(w, x, \lambda, u, 0)$, uniformly on U for fixed (w, x, λ), as $t \to +0$.*

Proof. We shall give the proof for a fixed σ, but with some detail, so that the continuity and uniformity assertions will be clear from Proposition 2.1.

Let $w^0 = w_0 = w$, $x^0 = x_0 = x$ and $e_0 = (w_0, x_0, y_0)$, $e^0 = (w^0, x^0, y^0) \in \mathscr{D}_n$. Thus (2.4) holds for $m = 0$, so that by Proposition 2.2(a),

$$\| R_n(w, x, y^0) - R_n(w, x, y_0) \| \geqslant \alpha^n \| y^0 - y_0 \|. \qquad (2.12)$$

Thus, for fixed (w, x), the equation

$$R_n(w, x, y) = 0 \tag{2.13}$$

has at most one solution y. Define the set

$$J_n = \{(w, x) : (2.13) \text{ has a solution } y, (w, x, y) \in \mathscr{D}_n\}.$$

In particular $(w, x) = 0 \in J_n \neq \varnothing$.

We shall show, by the implicit function theorem, that J_n is open. Let $(w_0, x_0) \in J_n$ and $R_n(w_0, x_0, y_0) = 0$. By (2.12), $D_y R_n(w, x, y)$ is invertible if $(x, y) \in \mathscr{D}_n$, also $\|[D_y R_n]^{-1}\| \leqslant 1/\alpha^n$. Put $D_0 = D_y R_n(w_0, x_0, y_0)$ and write (2.13) as $y = D_0^{-1} R_n(w, x, y) = y$, so that a solution y of (2.13) is a fixed point of the map $y \mapsto y - D_0^{-1} R_n(w, x, y) \equiv C_{w,x}(y)$, depending on parameters w, x. Since $D_y C_{w,x} = 0$ at (w_0, x_0, y_0), $\| D_y C_{w,x} \| \leqslant \theta < 1$ for (w, x, y) near (w_0, x_0, y_0). It follows that there are positive numbers ϵ, s such that, for fixed $(w, x) \in W_s(w_0) \times X_s(x_0)$, $C_{w,x}$ is a contraction mapping of $Y_\epsilon(y_0)$ into $Y_\epsilon(y_0)$, with contracting factor θ. Hence, $W_s(w_0) \times X_s(x_0) \in J_n$, and J_n is open.

Let $(w, x) \in J_n$, so that $e_{0n}(w, x) = (w, x, y_{0n}(w, x)) \in \mathscr{D}_n$ and put $e_{kn}(w, x) = S_n(e_{0n}(w, x))$ for $k = 0, 1, \ldots, n$. Thus, $y_{nn}(w, x) = R_n(e_{0n}(w, x)) = 0$; so that if (w^0, x^0), $(w_0, x_0) \in J_n$, Proposition 2.2(b) gives

$$\begin{aligned} &\| y_{kn}(w^0, x^0) - y_{kn}(w_0, x_0)\| \\ &\qquad \leqslant (1 - 2\delta)(\delta \| w^0 - w_0 \| + \| x_{kn}(w^0, x^0) - x_{kn}(w_0, x_0)\|), \end{aligned} \tag{2.14}$$

$$\| x_{kn}(w^0, x^0) - x_{kn}(w_0, x_0)\| \leqslant \delta \| w^0 - w_0 \| + (a + \delta^2)^k \| x^0 - x_0 \|, \tag{2.15}$$

for $k = 0, \ldots, n$. In particular, for $0 \leqslant k \leqslant n$ and $1 \leqslant m \leqslant n$,

$$\| x_{mn}(w, x)\| \leqslant \delta \| w \| + (a + \delta^2) \| x \| < (a + 2\delta) r < r, \tag{2.16}$$

$$\| y_{kn}(w, x)\| \leqslant (1 - 2\delta)(2\delta \| w \| + \| x \|) < (1 - 4\delta^2) r < r. \tag{2.17}$$

By the uniform continuity (2.14) of $y_{0n}(w, x)$ on J_n, $y_{0n}(w, x)$ and $e_0(w, x) = (w, x, y_{0n}(w, x))$ have unique continuous extensions to $\bar{J}_n \cap (WX)_r(0)$ satisfying (2.14)–(2.17). In particular, $e_0(w, x) \in \mathscr{D}_n$ and (2.11) holds on $\bar{J}_n \cap (WX)_r(0)$. Thus, J_n is both open and closed relative to $(WX)_r(0)$ and so, $J_n = (WX)_r(0)$. This completes the proof of Proposition 2.3.

PROPOSITION 2.4. *Let $y_{0n}(w, x)$ be given by Proposition 2.3. Then*

$$y_0(w, x) = \lim_{n \to \infty} y_{0n}(w, x) \textit{ exists uniformly} \tag{2.18}$$

on $(WX)_r(0) \times \Sigma$. *Also, the stability set* (1.8) *satisfies*

$$\mathscr{D} \supset \{(w, x, y) : y = y_0(w, x) \text{ on } (WX)_r(0)\}, \tag{2.19}$$

and (2.1), (2.2) *hold.*

Proof. Use the notation of the last proof, and let $1 \leqslant k \leqslant n$. Since $w_{0n}(w, x) = w_{0k}(w, x) = w$ and $x_{0n}(w, x) = x_{0k}(w, x) = x$, Proposition 2.2(a), with $m = 0$ in (2.4), and $y_{kk} = 0$ give

$$r > \| y_{kn} \| = \| y_{kn} - y_{kk} \| \geqslant \alpha^k \| y_{0n}(w, x) - y_{0k}(w, x) \|.$$

This proves the statement concerning (2.18), since $\alpha > 1$.

Keeping k fixed and letting $n \to \infty$ in (2.14), (2.15) gives (2.1), (2.2), hence $\| y_k \| \leqslant (1 - 4\delta^2)r < r$ for $k \geqslant 0$, and $\| x_k \| \leqslant (a + 2\delta)r < r$ for $k \geqslant 1$. This implies (2.19).

PROPOSITION 2.5. *The relations* (1.9) *and* (2.3) *hold.*

Proof. On (1.9). It only remains to prove the reverse inclusion to (2.19). Suppose that $(w, x, y) \in \mathscr{D}$ and $y \neq y_0(w, x)$. Let $e_{kn}(w, x) = S_k(w, x, y_{0n}(w, x))$ for $k = 0,..., n$ and $e_k(w, x) = S_k(w, x, y)$ for $k = 0, 1,...$. Since $w_{0n} = w_0 = w$, $x_{0n} = x_0 = x$, Proposition 2.2(a) and $y_{nn} = 0$ give $\| y_n \| = \| y_{nn} - y_n \| \geqslant \alpha^n \| y_{0n}(w, x) - y \| \sim \alpha^n \| y_0(w, x) - y \| \to \infty$, as $n \to \infty$. This contradicts $(w, x, y) \in \mathscr{D}$.

On (2.3). In view of what has been proved, for every $(w, x) \in (WX)_r(0)$, there is one and only one y such that $e = (w, x, y)$ is in the stability set $\mathscr{D}(\{T_n\}, E_r(0))$. The relation (2.3) follows by applying this uniqueness statement to the sequence $T_k, T_{k+1}, ...$.

3. SMOOTHNESS OF STABLE MANIFOLDS

Let $e_0(w_0, x_0) = (w_0, x_0, y_0(w_0, x_0)) \in \mathscr{D}$; $e_n(w_0, x_0) = S_n(e_0)$ for $n = 1, 2,...$; and

$$\Omega = L(W, W), \qquad \Xi = L(X, X), \qquad H = L(W \times X, Y) \tag{3.1}$$

Banach spaces of bounded linear operators. Define a sequence of linear maps of $\Omega\Xi H$ into itself by

$$\begin{aligned} &\tau_{n0}(\omega, \xi, \eta) \\ &\quad = (K_{n0}\omega, A_n\xi + M_{n0}\omega + N_{n0}\xi + O_{n0}\eta, B_n\xi + P_{n0}\omega + Q_{n0}\xi + R_{n0}\eta), \end{aligned} \tag{3.2}$$

where $n = 1, 2,\ldots$ and

$$\begin{aligned} K_{n0} &= D_w U_n(w_{n-1}(w_0, x_0)); \\ M_{n0}, N_{n0}, O_{n0} &= D_w, D_x, D_y F_n(e_{n-1}(w_0, x_0)); \\ P_{n0}, Q_{n0}, R_{n0} &= D_w, D_x, D_y G_n(e_{n-1}(w_0, x_0)); \end{aligned} \tag{3.3}$$

so that τ_{n0} depends on parameters (w_0, x_0, σ). By Lemma 2.1, there exists a function

$$\eta = \gamma(\omega, \xi) = \gamma(\omega, \xi, w_0, x_0, \sigma) \tag{3.4}$$

in $C^0(\Omega\Xi \times (WX)_r(0) \times \Sigma)$, such that for a fixed (w_0, x_0, σ) and $\rho > 0$, the restriction $\gamma \mid (\Omega\Xi)_\rho(0)$ gives the stable manifold $\mathscr{D}(\{\tau_{n0}\}, (\Omega\Xi H)_\rho(0))$.

THEOREM 3.1. *Under assumption* (H), *the stability set* $\mathscr{D}(\{T_n\}, E_r(0))$ *is a manifold* $y = y_0(w, x) = y_0(w, x, \sigma)$, *where* y_0 *satisfies the conclusions of Lemma* 2.1(a) *and has a Fréchet derivative* $D_{(w,x)} y_0$, *continuous in* (w, x, σ), *given by*

$$D_{(w,x)} y_0 = \gamma(id_W, id_X, w, x, \sigma). \tag{3.5}$$

This will be deduced from Lemma 2.1 in the next section. In the case (1.0) or equivalently (1.1), we can obtain the existence of derivatives of y_0 with respect to some of the parameters on which it depends, and also higher order differentiability. In the next theorem, we assume that the parameter space Σ is of the form

$$\sigma = (v, \lambda) \in \Sigma = V^0 \times \Lambda, \tag{3.6}$$

where V^0 is an open ball in a Banach space V and Λ is a metric space. We shall also assume that

$$U_n(w) = K_n w, \qquad K_n \in L(W, W), \qquad \| K_n \| \leqslant 1, \tag{3.7}$$

$$K_n = K_n(\lambda) \text{ does not depend on } v. \tag{3.8}$$

THEOREM 3.2_k. *Assume* (H), (3.6), (3.7), *and* (3.8). *Suppose that the map* T_n *in* (1.2) *is of class* C^k, $k \geqslant 1$, *as function of* (w, x, y, v), *that the derivatives of* A_n, B_n, F_n, G_n *are continuous functions of* (w, x, y, v, λ), *that the first order derivatives are bounded on* {*small* v-*balls*} $\times E_s(0) \times \Lambda \times \{n = 1, 2,\ldots\}$ *for all* s, $0 < s < r$, *and that all derivatives of order* $\leqslant k$ *are bounded on* {*small* (v, w, x, y)-*balls*} $\times \Lambda \times \{n = 1, 2,\ldots\}$. *Then* $y_0(w, x, v, \lambda)$ *is of class* C^k *with respect to* (w, x, v) *and its derivatives are continuous functions of* (w, x, v, λ).

Furthermore, if $e_0 = (w, x, y_0(w, x, v, \lambda)) \in E$ *and* $e_n = S_n(e_0)$ *for* $n = 1, 2, \ldots$, *then the first order derivatives* $D_{(w,x,v)}e_n$ *are bounded on*

$$\{\textit{small } v\textit{-balls}\} \times E_s(0) \times \Lambda \times \{n = 0, 1, \ldots\}, \qquad 0 < s < r,$$

and all derivatives of order $\leqslant k$ *are bounded on*

$$\{\textit{small } (w, x, v)\textit{-balls}\} \times \Lambda \times \{n = 0, 1, \ldots\}.$$

4. Proof of Theorem 3.1

Let $\epsilon > 0$ be arbitrarily small and $s = r - \epsilon$. We shall prove Theorem 3.1 on $E_s(0)$. From now on, assume that $(w_0, x_0) \in (WX)_s(0)$ and $(w, x) \in (WX)_1(0)$, so that $(w_0 + tw, x_0 + tx) \in (WX)_r(0)$ if $t \in T = [0, \epsilon]$. Using the notations of Section 2, let $e_0 = (w_0, x_0, y_0(w_0, x_0))$ and $e_n = S_n(e_0)$ for $n = 1, 2, \ldots$. For $0 < t \leqslant \epsilon$, put

$$\bar{e}_n \equiv (\bar{w}_n, \bar{x}_n, \bar{y}_n) = [e_n(w_0 + tw, x_0 + tx) - e_n(w_0, x_0)]/t. \qquad (4.1_n)$$

By (2.1), (2.2) in Lemma 2.1,

$$\| \bar{e}_n \| \leqslant 1 \qquad \text{for} \quad n = 0, 1, \ldots. \qquad (4.2)$$

We can write

$$\bar{e}_n = T_n'(\bar{e}_{n-1}), \qquad (4.3)$$

where T_n' is the linear map

$$\begin{aligned} &T_n'(\bar{w}, \bar{x}, \bar{y}) \qquad (4.4)\\ &\quad = (K_n\bar{w}, A_n\bar{x} + M_n\bar{w} + N_n\bar{x} + O_n\bar{y}, B_n\bar{y} + P_n\bar{w} + Q_n\bar{x} + R_n\bar{y}), \end{aligned}$$

in which

$$\begin{aligned} K_n = \int_0^1 D_w U_n \, d\theta; \qquad M_n, N_n, O_n &= \int_0^1 D_w, D_x, D_y F_n \, d\theta; \\ P_n, Q_n, R_n &\int_0^1 D_w, D_x, D_y G_n \, d\theta; \end{aligned} \qquad (4.5)$$

and the argument of the integrands is

$$\theta e_n(w_0 + tw, x_0 + tx) + (1 - \theta) \, e_n(w_0, x_0).$$

Although $t > 0$ is required in (4.1), (4.2), we allow $t = 0$ in (4.5); so that the operators there do not depend on (w, x) at $t = 0$ and tend, as $t \to +0$, to their values (3.3) at $t = 0$ uniformly for $(w, x) \in (WX)_1(0)$, with (w_0, x_0, σ) fixed.

By Lemma 2.1, there is a function

$$\bar{y} = h(\bar{w}, \bar{x}) = h(\bar{w}, \bar{x}, w_0, x_0, \sigma, w, x, t)$$

defined on

$$WX \times (WX)_s(0) \times \Sigma \times (WX)_1(0) \times [0, \epsilon]$$

such that, for a fixed set of parameters,

$$\mathscr{D}(\{T_n'\}, E_\rho(0)) = \{(\bar{w}, \bar{x}, \bar{y}) : \bar{y} = h(\bar{w}, \bar{x}) \text{ on } (WX)_\rho(0)\}$$

for all $\rho > 0$.

One can associate with (4.4) a linear map of $\Omega\Xi H$ into itself,

$$\tau_n(\omega, \xi, \eta) = (K_n\omega, A_n\xi + M_n\omega + N_n\xi + O_n\eta, B_n\eta + P_n\omega + Q_n\xi + R_n\eta). \tag{4.6}$$

Again, by Lemma 2.1, there is a function $\eta = \gamma(\omega, \xi)$, defined on $\Omega\Xi$ and depending continuously on parameters $(w_0, x_0, \sigma, w, x, t)$, such that the stable manifold $\mathscr{D}(\{T_n\}, (\Omega\Xi H)_\rho(0))$ is $\eta = \gamma \mid (\Omega\Xi)_\rho(0)$ for any $\rho > 0$. At $t = 0$, (4.6) reduces to (3.2), (3.3) and is independent of (w, x). By Lemma 2.1,

$$\eta = \gamma(\omega, \xi) = \gamma(\omega, \xi, \omega_0, x_0, \sigma, w, x, t),$$

as an $H = L(WX, Y)$-valued function, is continuous in all of its variables and, as $t \to +0$,

$$\gamma(\omega, \xi, \omega_0, x_0, \sigma, w, x, t) \to \gamma(\omega, \xi, w_0, x_0, \sigma) = \gamma(\omega, \xi, w_0, x_0, \sigma, \omega, x, 0)$$

uniformly for $(w, x) \in (WX)_1(0)$, for fixed $(\omega, \xi, w_0, x_0, \sigma)$. In particular, if $(\omega, \xi) = (id_W, id_X)$ is fixed, then

$$\gamma(id_W, id_X, w_0, x_0, \sigma, w, x, t)(\bar{w}, \bar{x}) \to \gamma(id_W, id_X, w_0, x_0, \sigma)(\bar{w}, \bar{x}), \tag{4.7}$$

as $t \to +0$, uniformly for $(w, x), (\bar{w}, \bar{x}) \in (WX)_1(0)$, and fixed (w_0, x_0, σ).

On the one hand, it is clear that

$$h(\bar{w}, \bar{x}, w_0, x_0, \sigma, w, x, t) = \gamma(id_W, id_X, w_0, x_0, \sigma, w, x, t)(\bar{w}, \bar{x}). \tag{4.8}$$

On the other hand, (4.2), (4.3) imply that $\bar{e}_0 \in \mathscr{D}(\{T_n'\}, E_\rho(0)\}$ for $\rho > 1$; so that, by (4.1_0),

$$h(w, x, w_0, x_0, \sigma, w, x, t) = [y_0(w_0 + tw, x_0 + tx) - y_0(w_0, x_0)]/t.$$

Thus, (4.7) and (4.8) give, as $t \to +0$,

$$[y_0(w_0 + tw, x_0 + tx) - y_0(w_0, x_0)]/t \to \gamma(id_W, id_X, w_0, x_0, \sigma)(w, x)$$

uniformly for $(w, x) \in (WX)_1(0)$, for fixed (w_0, x_0, σ). This proves Theorem 3.1.

5. Proof of Theorem 3.2_1

Since the assertions to be proved (that is, the existence and continuity of $D_{(w,x,v)}\, y_0$ near a point (w^1, x^1, v^1)) are local, we can restrict our attention to v near v^1. It will also be clear that the only $e = (w, x, y)$ which come into consideration are $e \in E_s(0)$, where s ($<r$) is chosen so that $E_s(0)$ contains all of the iterates $e_n = e_n(w, x)$, $n = 1, 2,\ldots,$ for (w, x, v) near (w^1, x^1, v^1). Thus we can suppose that there is a constant C such that

$$\| D_v A_n \|, \| D_v B_n \|, \| D_v F_n \|, \| D_v G_n \| \leqslant C \tag{5.1}$$

on $E_s \times V^0 \times \Lambda$. By replacing v by a new variable $v^1 + \epsilon v$, we can suppose that C is arbitrarily small, that V^0 is centered at 0. We can also suppose that $V^0 = V_s(0)$.

In what follows, we consider the space W to be replaced by the space VW, and deal with a sequence of maps $T_{11}, T_{21},\ldots,$ depending on a parameter λ, from $(VE)_s(0)$ into $VE = VWXY$,

$$T_{n1}(v, e) = (v, K_n w, A_{n1} x + F_{n1}, B_{n1} y + G_{n1}), \tag{5.2}$$

where the linear maps K_n, $A_{n1} = A_n(0, \lambda)$, and $B_{n1} = B_n(0, \lambda)$ depend only on λ, and

$$F_{n1}(v, e, \lambda) = F_n(e, v, \lambda) + [A_n(v, \lambda) - A_n(0, \lambda)]x,$$
$$G_{n1}(v, e, \lambda) = G_n(e, v, \lambda) + [B_n(v, \lambda) - B_n(0, \lambda)]y.$$

Note that the linear map $(v, w) \to (v, K_n w)$ has a norm 1, and that a Lipschitz constant for F_{n1}, G_{n1} with respect to (v, e) is $\delta^2 + 2C + Cs \equiv \delta'^2$. Since C is arbitrarily small, we can suppose that the inequalities (1.4) hold if δ is replaced by $\delta' > 0$.

It follows from Theorem 3.1 that the set $\mathscr{D}(\{T_{n1}\}, (VE)_s(0))$ is a manifold $y = y_0(w, x, v) = y_0(w, x, v, \lambda)$, $(w, x, v) \in (VWX)_s(0)$, and that $D_{(w,x,v)}\, y_0$ exists and is a continuous function of (w, x, v, λ). By the analogues of (2.1), (2.2), we also have that if $e_0 = (w, x, y_0(w, x, v))$ and $e_n = T_n(e_{n-1})$, so that $(v, e_n) = T_{n1}(v, e_{n-1})$, then $\| D_{(w,x,v)} e_n \| \leqslant 1$ on $(VE)_s(0) \times \Lambda$. This proves Theorem 3.2_1.

6. Proof of Theorem 3.2_k, $k > 1$

Let $k > 1$ and Theorem 3.2_{k-1} hold. For a moment, assume that no parameters v occur in T_n. Consider the map (3.2), (3.3), where $K_{n0} = K_n$ depends only on $\sigma = \lambda$. The map τ_{n0} depends on parameters (w_0, x_0, λ) and, on any given sphere $(\Omega\Xi H)_\rho(0)$, satisfies the conditions of Theorem 3.2_{k-1}, in which (w_0, x_0) plays the role of the parameter v. Thus, by Theorem 3.2_{k-1}, the function (3.4) is of class C^{k-1} with respect to (ω, ξ, w_0, x_0) with derivatives continuous in $(\omega, \xi, w_0, x_0, \lambda)$ and the specified boundedness properties. Theorem 3.2_k then follows, for the case under consideration, from (3.5).

In the case that parameters v do occur in T_n, apply the same arguments to the maps belonging to (5.2) in the same way that (3.2)-(3.3) belongs to (1.2). This completes the proof.

7. Remarks

The conditions and arguments above were designed to yield a simple proof of the C^k, $k \geqslant 1$, result Theorem 3.2_k. If the object is merely to obtain a C^1-result when no parameters w occur, the "hyperbolicity" conditions can be replaced by those in the finite dimensional case (cf. [3], Lemma 5.1, pp. 234-235]) and the proofs simplified.

Hypothesis (H_1). Let $T_n(z) = (A_n x, B_n y) + (F_n(z), G_n(y))$ be a map from $Z_r(0)$ to Z having a continuous Fréchet derivative; $A_n \in L(X, X)$, $B_n \in L(Y, Y)$; B_n is invertible;

$$\begin{gathered} \| A_n \| \leqslant a < 1, \qquad \| B_n^{-1} \| \leqslant 1/b, \\ 0 < 4\delta < b - a \qquad \text{and} \qquad a + 2\delta < 1 \end{gathered} \tag{7.1}$$

(so that $b > 1$ is not assumed); finally, $F_n(0) = 0$, $G_n(0) = 0$, and

$$\| F_n(z_1) - F_n(z_2) \|, \qquad \| G_n(z_1) - G_n(z_2) \| \leqslant \delta \| z_1 - z_2 \|. \tag{7.2}$$

Introduce the set

$$\mathscr{D}^{a\delta} = \{ z_0 = (x_0, y_0) \in \mathscr{D} : z_n \equiv S_n(z_0) \equiv (x_n, y_n) \text{ satisfy (7.4)} \}, \tag{7.3}$$

$$\| y_n \| \leqslant \| x_n \| \leqslant (a + 2\delta)^n \| x_0 \| \qquad \text{for} \quad n = 0, 1, \ldots. \tag{7.4}$$

It is not difficult to see (from Proposition 7.1 below) that when $z_0 \in \mathscr{D}$, condition (7.4) is equivalent to

$$\| z_n \| = \max(\| x_n \|, \| y_n \|) = O(\theta^n) \quad \text{for some } \theta, 0 < \theta < \min(1, b - 2\delta). \tag{7.5}$$

It is clear that if the analogue of condition (7.2) holds for arbitrarily small $\delta > 0$ when z_1, z_2 are in a ball $Z_n(0)$ of sufficiently small radius $\eta > 0$, then $\mathscr{D}^{a\delta}$ does not depend on δ and (7.4) is equivalent to

$$\limsup_{n\to\infty} n^{-1} \log \| z_n \| \leqslant a. \tag{7.6}$$

(Also, we have that $\mathscr{D}^{a\delta} = \mathscr{D}$ if $b - 2\delta > 1$.)

THEOREM 7.1. *Under the assumption* (H_1), *the set* $\mathscr{D}^{a\delta}$ *(for fixed* $\sigma \in \Sigma$*) is a* C^1*-manifold* $y = y_0(x) = y_0(x, \sigma)$, *where* $y_0(x, \sigma)$ *is continuous and has a continuous Fréchet derivative* $D_x y_0$ *on* $X_r(0) \times \Sigma$, *and* $z_n = S_n(x_0, y_0)) \equiv (x_n(x_0), y_n(x_0))$ *satisfies*

$$\| y_n(x^0) - y_n(x_0)\| \leqslant \| x_n(x^0) - x_n(x_0)\| \leqslant (a + 2\delta)^n \| x^0 - x_0 \|, \tag{7.7}$$

$$y_n(x_0) = y_0(x_n(x_0)), \tag{7.8}$$

for $| x_0 |, | x^0 | < r$ *and* $n = 0, 1, \ldots$. *The Fréchet derivative* $D_x y_0$ *is given by the analogue of* (3.5) *where* W, Ω, w, ω *do not occur in* (3.1)-(3.5).

The proof follows that of Theorem 3.1. One introduces an hypothesis (H_{10}), which is the analogue of (H_0) in which w does not occur, and proves an analogue of Lemma 2.1. In the proof of this lemma, the role of the main Proposition 2.2 is played by the simpler

PROPOSITION 7.1. *Let* $n > 0$; $z_0, z^0 \in \mathscr{D}_n = \mathscr{D}(S_n)$; *and* $z_k = S_k(z_0)$, $z^k = S_k(z^0)$ *for* $k = 0, \ldots, n$. (a) *The inequality*

$$\| y^m - y_m \| \geqslant \| x^m - x_m \| \qquad \text{for some } m,\ 0 \leqslant m < n, \tag{7.9}$$

implies that, for $k = m + 1, \ldots, n$,

$$\| y^k - y_k \| \geqslant \| x^k - x_k \|, \| y^k - y_k \| \geqslant (b - 2\delta)^{k-m} \| y_m - y_m \|. \tag{7.10}$$

(b) *The inequality*

$$\| y^n - y_n \| < \| x^n - x_n \| \tag{7.11}$$

implies that, for $k = 0, 1, \ldots, n$,

$$\| y^k - y_k \| \leqslant \| x^k - x_k \| \leqslant (a + 2\delta)^k \| x^0 - x_0 \|. \tag{7.12}$$

After one obtains the analogue of Lemma 2.1, the proof of Theorem 7.1 is the same as that of Theorem 3.1, except that w and ω do not occur and (4.2) is replaced by

$$\| \bar{y}_n \| \leqslant \| \bar{x}_n \| \leqslant (a + 2\delta) \| \bar{x}_0 \|, \qquad \bar{x}_0 = x.$$

APPENDIX. Stable Manifolds of Anosov Diffeomorphisms[1]

Since all the tools are at hand, it is perhaps worthwhile to collect, in one place, statements and proofs of the properties, other than measure theoretical properties, of the stable and unstable manifolds of an Anosov diffeomorphism used, for example, in the proof of ergodicity [A1]. (For remarks on this proof, see the end of this Appendix.)

Notation. Let $M = M^d$ be a C^∞, connected, d-dimensional, compact, Riemann manifold (without boundary). $T_m(M)$ denotes the tangent space of M at the point $m \in M$. If $\phi : M \to M$ is a C^1 map, ϕ^* denotes the induced tangent map $\phi^* : T(M) \to T(M)$, so that

$$\phi_m{}^* = \phi^* \mid T_m(M) : T_m(M) \to T_{\phi(m)}(M).$$

The distance between points $m, p \in M$ will be denoted by $| m, p |$.

Definition. A diffeomorphism $\phi : M \to M$ of class C^1 is called an Anosov diffeomorphism if there exist constants, θ_0 and c_0, $0 < \theta_0 < 1$ and $c_0 \geqslant 1$, such that, for every $m \in M$, the tangent space $T_m(M)$ is a direct sum $T_m(M) = X_m{}^k \oplus Y_m{}^j$, $j + k = d$, where $x \in X_m{}^k$ and $y \in Y_m{}^j$ imply

$$| \phi^{*n}x | \leqslant c_0\theta_0{}^n \mid x \mid, \quad | \phi^{*n}y | \geqslant | y |/c_0\theta_0{}^n \quad \text{for} \quad n \geqslant 0,$$

$$| \phi^{*n}x | \geqslant | x |/c_0\theta_0{}^{-n}, \quad | \phi^{*n}y | \leqslant c_0\theta_0{}^{-n} | y | \quad \text{for} \quad n \leqslant 0.$$

Remark. Except for a change of constants, c_0 and θ_0, this definition does not depend on the choice of the Riemann metric on M.

Definition. Let $\phi : M \to M$ be an Anosov diffeomorphism. The stable manifold $W_m{}^s$ of a point $m \in M$ is the set of points $p \in M$ such that

$$| \phi^n(m), \phi^n(p)| \to 0 \qquad \text{as} \quad n \to \infty.$$

The unstable manifold $W_m{}^u$ is the set of points p satisfying

$$| \phi^n(m), \phi^n(p)| \to 0 \qquad \text{as} \quad n \to -\infty.$$

It is clear that the set of stable and unstable manifolds are invariant under ϕ, that is, $\phi W_m{}^s = W^s_{\phi(m)}$ and $\phi W_m{}^u = W^u_{\phi(m)}$.

Remark. For any integer $N > 0$, the stable and unstable manifolds of ϕ and ϕ^N are identical. Hence, in what follows, we can suppose that $c_0 = 1$; for otherwise, we replace ϕ by ϕ^N, c_0 by 1, and θ_0 by $c_0\theta^N$ (<1).

[1] Added June 5, 1970.

Distinguished Charts and the Maps T_n

At any point $m \in M$, we have a chart $(\exp_m^{-1}, U_\rho)$, where $\exp_m : B_m(\rho) \to U_\rho$ is a diffeomorphism, $B_m(\rho)$ is the ball of radius ρ in $T_m(M)$. If $\xi \in T_m(M)$, we have the unique decomposition, $\xi = x + y$ for $x \in X_m{}^k$, $y \in Y_m{}^j$ and we replace the local coordinate ξ by (x, y) for the point $p = \exp_m \xi = \exp_m(x + y) \equiv \sigma_m(x, y)$. We also replace the neighborhood U_ρ of M by the image $U_m = U_m(r)$ of the product of the spheres $\{|x| < r\} \times \{|y| < r\}$. The number $r > 0$ can be chosen independent of m. In what follows, we deal only with charts (σ_m^{-1}, U_m) and local coordinates (x, y).

Associated with ϕ and a point $m \in M$, we define a sequence of maps $T_1, T_2, \ldots$

$$T_n : \{|x| < r\} \times \{|y| < r\} \to \{|x| < s\} \times \{|y| < s\} \tag{A.1}$$

given by

$$T_n = \sigma^{-1}_{\phi^n_{(m)}} \circ \phi \circ \sigma_{\phi^{n-1}_{(m)}} \qquad \text{for } n = 1, 2, \ldots . \tag{A.2}$$

Let S_n denote

$$S_n = T_n \circ \cdots \circ T_1 = \sigma^{-1}_{\phi^n_{(m)}} \circ \phi \circ \sigma_m . \tag{A.3}$$

The numbers $0 < r < s$ can be chosen independent of m, in such a way that the definition (A.1) is meaningful.

Since $c_0 = 1$ and $0 < \theta_0 < 1$ do not depend on m, it is clear that if (A.1) is identified with the case of (1.2), where the variable w does not occur, then the analogue of conditions (1.3)–(1.7) and (1.10)–(1.12) hold (with r, s, a, b, δ independent of $m \in M$ and $n = 1, 2, \ldots$).

Below, we also use (A.1), (A.2) with $n = 0, -1, -2, \ldots$. The number $r > 0$ is sufficiently small but independent of any choice of points $m \in M$.

We now list the desired results about stable manifolds. Those for unstable manifolds are, of course, completely analogous and obtained by replacing ϕ by ϕ^{-1}.

PROPOSITION 1. *$X_m{}^k$ is continuous with respect to $m \in M$ and $k = \dim X_m{}^k$ is independent of m.*

PROPOSITION 2. *$X_m{}^k$ is uniformly Hölder continuous on M if $\phi \in C^2$.*

PROPOSITION 3. *Locally, the stable manifold $W_m{}^s$ of m is a k-dimensional submanifold of M of class C^1 and is the unique k-dimensional integral manifold through m of the field $X_m{}^k$.*

PROPOSITION 4. *The stable manifolds $W_m{}^s$ are "uniformly C^1 continuous" in the following sense: In a local chart (σ_m^{-1}, U_m), a piece of $W_m{}^s$ has an equation $y = y(x) = y(x, m)$ for $|x| < r/2$ with $y(0) = 0$; the function $y(x)$ is of class C^1, and y and its first order derivatives with respect to x are equicontinuous functions of (x, m) for $|x| < r/2$ and $m \in M$.*

This proposition can be strengthened as follows:

PROPOSITION 5. *There exist constants $c_1 > 0$, θ_1 $(0 < \theta_1 < 1)$ with the following properties: Consider an arbitrary local chart $(\sigma_p^{-1}, U_p(r))$ of M and a piece of a flat $\Pi = \{(x, y) : y = y_0, |x| < r\}$ in the tangent space $T_p(M)$, where y_0 is a constant, $|y_0| < r$. Let $n > 0$, $W_n = \phi^{-n}\sigma_p\Pi$, $m \in W_n \subset M$, and $W_n(m)$ the connected component of $W_n \cap U_m(r/2)$ containing m. Then the manifold $\sigma_m^{-1}W_n(m) \subset T_m(M)$ has an equation $y = y_{npm}(x)$ for $|x| < r/2$ such that the C^1-distance between $y = y_{npm}(x)$ and the function $y = y(x) = y(x, m)$ of Proposition 4 for $|x| < r$ is at most $c_1\theta_1{}^n$.*

Proposition 1 is contained in Anosov [A2, pp. 6–7]. Proposition 2 concerning Hölder continuity is given in Anosov [A3]. (In [A2], he also shows that $X_m{}^k$ need not be of class C^1 even if $\phi \in C^\infty$.) Propositions 3, 4, and 5 have analogues if $\phi \in C^i$, $i \geqslant 1$, obtained by replacing C^1 there by C^i.

On Proposition 1. Let $m(1), m(2),... \in M$, $\xi_n \in X^k_{m(n)}$, and $(m(n), \xi_n) \to (m, \xi) \in T_m(M)$ as $n \to \infty$. The continuity of ϕ^{*n}, for fixed n, shows that $\xi \in X_m{}^k$, so that $\limsup X^k_{m(n)} \subset X_m{}^k$. Similarly, $\limsup Y^j_{m(n)} \subset Y_m{}^j$. Since $k(m) + j(m) = d$ for all m, it is clear that $\lim X^k_{m(n)}$, $\lim Y^j_{m(n)}$ exist, and are $X_m{}^k$, $Y_m{}^j$. This implies Proposition 1.

On Proposition 2. Consider a chart (σ_m^{-1}, U_m) at m. The map $(\sigma_m{}^*)^{-1}$ maps the tangent space of U_m into that of $\Sigma = \{|x| < r\} \times \{|y| < r\}$. For $p \in U_m$, let $\bar{X}_p{}^k = (\sigma_m{}^*)^{-1}(X_p{}^k)$. We denote a vector in the tangent space $T_{(x,y)}(\Sigma)$ by $\zeta = (\xi, \eta)$. In particular, if $(x, y) = (0, 0)$, then $(\xi, \eta) \in \bar{X}_m{}^k$ if and only if $\eta = 0$. It follows from the continuity of $X_m{}^k$ that if r is sufficiently small (independent of m), then $0 \neq \zeta = (\xi, \eta) \in \bar{X}_p{}^k$ implies that $|\eta| \leqslant |\xi|$. For $p \in U$, put

$$D(p, m) = \sup |\eta|/|\xi| \qquad \text{for} \quad 0 \neq \zeta = (\xi, \eta) \in X_p{}^k,$$

so that $D(p, m) + |p, m|$ is essentially the distance from $X_p{}^k$ to $X_m{}^k$ in $T(M)$. For small δ, put

$$\omega(\delta) = \sup(D(p, m) + |p, m|) \qquad \text{for} \quad |m, p| \leqslant \delta. \tag{A.4}$$

Under the map $T_0{}^*$, the linear space $\bar{X}_p{}^k$ is mapped onto $\bar{X}^k_{\phi^{-1}(p)}$. There exist θ_1 and θ_2, $0 < \theta_2 < \theta_1 < 1$, such that if $\zeta = (\xi, \eta) \neq 0$ is in $T_{(x,y)}(\Sigma)$,

$|\eta| \leqslant |\xi|$, and $T_0{}^*(x, y; \zeta) = (x_1, y_1; \zeta_1)$, where $\zeta_1 = (\xi_1, \eta_1)$, then $|\eta_1|/|\xi_1| \leqslant \theta_1 |\eta|/|\xi| + |m, p|/\theta_2$. In fact, the action of $T_0{}^*$ on ζ is a linear map (depending uniformly Lipschitz continuously on (x, y)) such that $|\eta_1| \leqslant \theta_1 |\eta| + |m, p| . |\xi|/\theta_2$ and $|\xi_1| \geqslant |\xi|$. Consequently,

$$D(\phi^{-1}(m), \phi^{-1}(p)) \leqslant \theta_1 D(p, m) + |m, p|/\theta_2$$

or (A.5)

$$D(p, m) \leqslant \theta_1 D(\phi(p), \phi(m)) + |\phi(m), \phi(p)|/\theta_2.$$

Also, if θ_2 is sufficiently small,

$$0 < \theta_2 \leqslant |m, p|/|\phi(m), \phi(p)| \leqslant 1/\theta_2 . \tag{A.6}$$

Thus, we obtain

$$D(p, m) + |m, p| \leqslant \theta_1 D(\phi(m), \phi(p)) + 2|\phi(m), \phi(p)|/\theta_2 .$$

Take the supremum over the set $\{p: |m, p| \leqslant \delta\}$ to obtain

$$\omega(\delta) \leqslant \theta_1 \omega(\delta/\theta_2) + 2\delta/\theta_2{}^2$$

A simple argument shows that if $f(\delta) = \omega(\delta) + 2\delta/\theta_2(\theta_1 - \theta_2)$, then $f(\delta) \leqslant \theta_1 f(\delta/\theta_2)$ and there exists a constant C such that

$$\omega(\delta) \leqslant f(\delta) \leqslant C\delta^\alpha \tag{A.7}$$

for small $\delta > 0$, where $\alpha = \log \theta_1/\log \theta_2$.

This proves Proposition 2.

On Proposition 3. That, locally, $W_m{}^s$ is a C^1 k-dimensional submanifold of M follows from Theorem 3.1. It is clearly an integral manifold of the field $X_m{}^k$. The uniqueness assertion follows from the fact that if N is any integral manifold (of any dimension) of the field $X_m{}^k$ and $m(0)^i\in N$, then $N \subset W^s_{m(0)}$.

On Propositions 4 *and* 5. If, in the statement of Proposition 5, $p = \phi^n(m)$ and $y_0 = 0$, then the assertion of Proposition 5 is clear from the proofs of Lemma 2.1 and Theorem 3.1. These proofs also show that Proposition 5 is valid for any choice of p, y_0 and $m \in W_n$. Proposition 4 is an obvious corollary of Proposition 5.

Remarks on [A1]. It is proved in [A1] that an Anosov diffeomorphism of class C^2, with a finite invariant measure equivalent to volume on M, is ergodic. The use of the assumption $\phi \in C^2$ (in contrast to $\phi \in C^1$) is used in three places:

(1) to assure that ϕ^* is uniformly Lipschitz continuous on $T(M)$ in order to estimate certain Jacobian determinants (pp. 151–152);

(2) to assure the validity of Proposition 2;

(3) in Lemma 5.2 [A1, p. 153]. (Here the authors' considerations are not necessarily valid on a C^2 manifold V whose sectional curvatures need not be defined and Jacobi's equation may not exist unless, for example, V has a C^2 isometric immersion into a smooth manifold; cf., e.g., [A4, A5]. Actually, their Lemma 5.2 can be replaced by a much simpler assertion. In order to state the desired result, let us introduce some notations. Let $\omega(\delta)$ be a continuous, increasing function for $0 \leqslant \delta \leqslant \rho$ such that $\omega(0) = 0$. A complete Riemann manifold $V = V^k$ of dimension k will be said to be of class $\Omega_{\omega\rho}$ if it is of class C^1 and, for every $v \in V$ and δ, $0 < \delta \leqslant \rho$, there is a δ-ball $N_{v\delta}$ on V centered at v on which there are local coordinates $x : N_{v\delta} \to E^k$ such that $x(0) = v$, the element of arclength $ds^2 = g_{ij}(x)\, dx^i\, dx^j$ satisfies $(g_{ij}(0))$ is the unit matrix, and $| g_{ij}(x) - g_{ij}(x')| \leqslant \omega(| x - x' |)$ for $x, x' \in x(N_{v\delta}) \subset E^k$. An open set with a piecewise C^1 boundary will be called a curvilinear polyhedron.

LEMMA. *Corresponding to ω and ρ, there exist constants ϵ_0, r_0, A with the properties that if $V = V^k \in \Omega_{\omega\rho}$ and $G \subset V$ is an open set with a compact closure, then, for every r on $0 < r \leqslant r_0$, there exists a finite set of curvilinear polyhedra $\Pi_1, \ldots, \Pi_N$ in G such that*

(i) $\operatorname{diam} \Pi_i \leqslant 2r$;

(ii) $U(G, -4r) \subset \bigcup \Pi_i \subset U(G, -2r)$,

$$\text{where } U(G, -\epsilon) = \{v \in G : \operatorname{dist}(v, \partial G) > \epsilon\};$$

(iii) $\operatorname{meas} \Pi_i > r^k/A$; *and*

(iv) $A\epsilon r^k > \operatorname{meas}\{v \in V : \operatorname{dist}(v, \partial \Pi_i) < \epsilon r\}$ *for* $0 < \epsilon \leqslant \epsilon_0$.

Proof. This proof will only be indicated. It follows the first part of the proof of Lemma 5.2 [A1, pp. 153–158]. Let $v_1, \ldots, v_K$ be a maximal set of points in G with the property that $\operatorname{dist}(v_i, v_j) > r$ for $i \neq j$. Let

$$\Pi_i = \{v \in V : \operatorname{dist}(v, v_i) < \operatorname{dist}(v, v_j) \text{ for } j \neq i\}$$

and enumerate $v_1, \ldots, v_K$ so that $\operatorname{dist}(v_i, \partial G) > 3r$ for $i = 1, \ldots, N$ and $\operatorname{dist}(v_i, \partial G) \leqslant 3r$ for $i = N + 1, \ldots, K$. It is easy to verify that

$$\{v : \operatorname{dist}(v, v_i) < r/2\} \subset \Pi_i \subset \{v : \operatorname{dist}(v, v_i) \leqslant r\} \tag{A.9}$$

and that $\Pi_1, \ldots, \Pi_N$ satisfy (i)–(iii); cf., e.g., [A1, p. 153].

For $1 \leqslant i \leqslant N$ and $1 \leqslant j \leqslant K$, $j \neq i$, let

$$\Pi_{ij} = \{v \in V : r \geqslant \operatorname{dist}(v, v_i) = \operatorname{dist}(v, v_j)\}. \tag{A.10}$$

It is easy to see that there exist constants A and $r_0 > 0$, depending only on ω and ρ, such that for $0 < r \leqslant r_0$, the number of j, $1 \leqslant j \leqslant K$, for which $\Pi_{ij} \neq \varnothing$, does not exceed A. Hence, in verifying (iv), it suffices to show that, for a suitable choice of A, r_0, ϵ_0,

$$\operatorname{meas} \Gamma_{ij\epsilon} \leqslant A\epsilon r^k \qquad \text{for} \quad 0 < \epsilon \leqslant \epsilon_0, \tag{A.11}$$

where

$$\Gamma_{ij\epsilon} = \{v \in V : \operatorname{dist}(v, \Pi_{ij}) < \epsilon r\}. \tag{A.12}$$

From (A.9), (A.10), and (A.12), it is seen that if θ is defined by $1 + \theta = (1 + 2\epsilon)/(1 - 2\epsilon)$, then

$$\Gamma_{ij\epsilon} \subset \{v : \operatorname{dist}(v_j, v)/(1+\theta) \leqslant \operatorname{dist}(v_i, v) \leqslant (1+\theta) \operatorname{dist}(v_j, v) \leqslant (1 + 2\theta)r\}. \tag{A.13}$$

Hence, it is clear that (A.11) would follow if $V = E^k$, but then it is also clear that (A.11) follows from (A.13) if $V \in \Omega_{\omega\rho}$.

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